Atoms of complete intersections in flag varieties

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From curve counts to a birational obstruction

Can genus-zero curve counts tell us that a Fano fourfold is irrational?

GW invariants $\longrightarrow$ quantum multiplication

$\longrightarrow$ Hodge atoms $\longrightarrow$ birational obstructions

Genus-zero Gromov–Witten classes

Let Z be a smooth complex projective variety and $\beta \in \mathrm{NE}(Z) \subset \mathrm{H}_2(Z, \mathbb{Z})$ an effective curve class. The pair (n, β) is *stable* if $\beta \neq 0$ or $n \geq 3$.

Definition

The expected complex dimension is

$$\delta(n, \beta) = \dim Z + n - 3 + \int_\beta c_1(TZ).$$

The genus-zero Gromov–Witten class is

$$\mathrm{GW}_{0,n,\beta}(Z) \in \mathrm{H}_{2\delta(n,\beta)}(Z^n, \mathbb{Q}).$$

Stable maps and the virtual class

A stable map is a morphism $f : (C; x_1, \ldots, x_n) \to Z$ such that:

- C is a connected nodal projective curve of arithmetic genus 0;
- the markings are distinct and lie in the smooth locus of C;
- the automorphism group of the marked map is finite.

For $f_*[C] = \beta$, the moduli stack $\overline{\mathcal{M}}_{0,n}(Z, \beta)$ carries a virtual fundamental class

$$[\overline{\mathcal{M}}_{0,n}(Z, \beta)]^{\mathrm{vir}} \in \mathrm{H}_{2\delta(n,\beta)}(\overline{\mathcal{M}}_{0,n}(Z, \beta), \mathbb{Q}).$$

Evaluation maps and GW invariants

Each marking defines an evaluation map

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(Z,\beta) \to Z, \qquad [f : (C; x_1, \ldots, x_n) \to Z] \longmapsto f(x_i).$$

Put $\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_n)$. Then

$$\mathrm{GW}_{0,n,\beta}(Z) = \mathrm{ev}_*[\overline{\mathcal{M}}_{0,n}(Z,\beta)]^{\mathrm{vir}}.$$

For $\gamma_i \in \mathrm{H}^*(Z,\mathbb{Q})$, the numerical invariant is

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{0,\beta}^Z = \int_{[\overline{\mathcal{M}}_{0,n}(Z,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \gamma_i.$$

The genus-zero potential

Assume $\mathrm{H}^{\mathrm{odd}}(Z,\mathbb{Q}) = 0$, as in the paper's quantum-cohomology construction. Choose a homogeneous basis $T_0 = 1, T_1, \ldots, T_s$ of $\mathrm{H}^*(Z,\mathbb{Q})$ and put $\gamma = \sum_{i=0}^{s} y_i T_i$.

$$\Phi(\gamma) = \sum_{\beta \in \mathrm{NE}(Z)} {\sum_{n \geq 0}}' \frac{q^\beta}{n!} \langle \underbrace{\gamma, \ldots, \gamma}_{n} \rangle_{0,\beta}^{Z}.$$

The prime omits the unstable pairs $(\beta, n) = (0,0), (0,1), (0,2)$. The degree-zero part is the classical cubic potential:

$$\Phi_{\mathrm{cl}}(\gamma) = \frac{1}{6} \int_Z \gamma \smile \gamma \smile \gamma.$$

All terms with $\beta \neq 0$ are quantum corrections.

The divisor equation and Novikov variables

Let $T_1, \dots, T_r$ be a basis of divisor classes and write $\gamma = \gamma_{\mathrm{div}} + \gamma'$ with $\gamma_{\mathrm{div}} = \sum_{i=1}^{r} y_i T_i$. For $\beta \neq 0$, the divisor axiom yields

$$\Phi_{\mathrm{quant}}(\gamma) = \sum_{\substack{\beta \in \mathrm{NE}(Z) \\ \beta \neq 0}} q^{\beta} \exp\left(\int_{\beta} \gamma_{\mathrm{div}} \right) \sum_{n \geq 0} \frac{1}{n!} \langle \underbrace{\gamma', \dots, \gamma'}_{n} \rangle_{0,\beta}^{Z}.$$

The symbols q^{β} are formal Novikov monomials.

Big quantum multiplication

Let $g_{ij} = \int_Z T_i \smile T_j$ be the Poincaré pairing and $(g^{ij}) = (g_{ij})^{-1}$.

Definition

The big quantum cohomology and product are

$$Q\mathrm{H}^*(Z) = \mathcal{N}_Z[[y_0, \ldots, y_s]] \otimes_{\mathbb{Q}} \mathrm{H}^*(Z, \mathbb{Q}),$$

$$T_i \star T_j = \sum_{a,b=0}^{s} \frac{\partial^3 \Phi}{\partial y_i \, \partial y_j \, \partial y_a} \, g^{ab} T_b.$$

The WDVV equations imply that $\star$ is associative.

Quantum cohomology and the small locus

The Novikov ring is the completion

$$\mathcal{N}_Z = \mathbb{Q}\widehat{[\mathrm{NE}(Z)]}$$

along the ideal generated by nonzero effective classes.

Definition

The small quantum product is the specialization $y_i = 0$:

$$T_i \star_{\mathrm{small}} T_j = \sum_{\beta \in \mathrm{NE}(Z)} \sum_{a,b=0}^{s} q^\beta \langle T_i, T_j, T_a \rangle_{0,\beta}^Z \, g^{ab} T_b.$$

It makes $QH^*_{\mathrm{small}}(Z) = \mathcal{N}_Z \otimes_{\mathbb{Q}} H^*(Z, \mathbb{Q})$ a unital algebra.

Quantum multiplication and the spectrum

Input: Z smooth projective over $\mathbb{C}$ and the complete, algebraically closed non-Archimedean field $\Bbbk$ obtained by completing the Puiseux series over $\overline{\mathbb{Q}}$.

The A-model parameter space

The $\Bbbk$-analytic pre-F-manifold B_Z carries the quantum product $\star_b$ and a canonical identification

$$T_b B_Z \cong \big(\mathrm{H}^*(Z, \Bbbk), \star_b\big)$$

at each $\Bbbk$-point b.

The spectral operator

$$\boldsymbol{\kappa}_b := \mathrm{Eu}_b \star_b (-), \qquad \mathrm{spec}(\boldsymbol{\kappa}_b) = \{\text{distinct eigenvalues of } \boldsymbol{\kappa}_b\}.$$

The Euler field

At a $\Bbbk$-point $b \in B_Z(\Bbbk)$ the Euler vector field is

$$\mathrm{Eu}_b = c_1(TZ) + \frac{2 \cdot \mathbf{1} - \mathrm{Deg}}{2}(b), \qquad \mathrm{Deg}\big|_{\mathrm{H}^a(Z)} = a \ \mathrm{id}\,.$$

On the small quantum locus

When all non-Novikov parameters vanish, $\mathrm{Eu}_b = c_1(TZ)$, hence

$$\boldsymbol{\kappa}_b = c_1(TZ) \star_{\mathrm{small},b} (-).$$

How can its spectrum constrain birational geometry?

Iritani's blowup formula

For a disjoint union, the A-model data split:

$$B_{Z \sqcup W} \cong B_Z \times B_W, \qquad \mathrm{Eu}_{Z \sqcup W} = \mathrm{Eu}_Z \boxplus \mathrm{Eu}_W.$$

Now let $W \subset Z$ be smooth of codimension $c \geq 2$, and write

$$\widetilde{Z} = \mathrm{Bl}_W Z, \qquad E = \mathbb{P}(N_{W/Z}).$$

Introduce

$$Z' = Z \sqcup \underbrace{W \sqcup \cdots \sqcup W}_{c-1 \text{ copies}}.$$

The blow-up formula compares the A-model of $\widetilde{Z}$ with that of Z'.

Iritani's theorem

Theorem (Iritani; A-model consequence)

There are nonempty connected analytic domains $U_{\widetilde{Z}} \subset B_{\widetilde{Z}}$ and $U_{Z'} \subset B_{Z'}$, with an isomorphism of the corresponding pre-F-manifolds preserving Euler fields,

$$\Psi : U_{\widetilde{Z}} \xrightarrow{\sim} U_{Z'}.$$

At corresponding $\Bbbk$-points, its differential is an isomorphism of unital tangent algebras which intertwines multiplication by the Euler field.

Spectral decomposition

Definition

For $b \in B_Z(\mathbb{k})$, the generalized eigenspace at $\lambda \in \operatorname{spec}(\boldsymbol{\kappa}_b)$ is

$$E_{b,\lambda} := \ker(\boldsymbol{\kappa}_b - \lambda \operatorname{id})^N \qquad (N \gg 0).$$

Theorem (KKPY, local spectral decomposition)

Locally near b, the pre-F-manifold decomposes into spectral factors indexed by $\lambda \in \operatorname{spec}(\boldsymbol{\kappa}_b)$, with the corresponding Euler fields.

In particular,

$$T_b B_Z = \bigoplus_{\lambda \in \operatorname{spec}(\boldsymbol{\kappa}_b)} E_{b,\lambda}.$$

The Hodge group and its fixed locus

Equip $V = \mathrm{H}^*(Z, \mathbb{Q})$ with its graded polarizable Hodge structure. Let $\mathrm{MT}(V)$ be its Mumford–Tate group and let f be its Tate character. The Hodge group is

$$\mathrm{Hod} := \ker(f : \mathrm{MT}(V) \to \mathbb{G}_m)\,.$$

Its invariant vectors are precisely the rational Hodge classes $\mathrm{H}^*(Z)^{\mathrm{Hdg}}$.

The Hodge locus

At $b \in B_Z^{\mathrm{Hod}}(\Bbbk)$, the product $\star_b$ and the Euler field Eu_b are $\mathrm{Hod}_\Bbbk$-equivariant. Thus $\boldsymbol{\kappa}_b$ commutes with $\mathrm{Hod}_\Bbbk$ and each $E_{b,\lambda}$ is a $\mathrm{Hod}_\Bbbk$-subrepresentation.

Definition of coarse Hodge atoms

For $b \in B_Z^{\mathrm{Hod}}(\mathbb{k})$ let $D(b) = \#\operatorname{spec}(\boldsymbol{\kappa}_b)$, and write

$$\mathcal{V} = \left\{ b \in B_Z^{\mathrm{Hod}}(\mathbb{k}) : D(b) = \max_{a \in B_Z^{\mathrm{Hod}}(\mathbb{k})} D(a) \right\}.$$

Theorem (KKPY)

The locus $\mathcal{V}$ is nonempty and analytically open. For $b \in \mathcal{V}$, the representation class of $E_{b,\lambda}$ is independent of b and defined over $\overline{\mathbb{Q}}$.

Definition (Coarse Hodge atom)

A coarse Hodge atom is the isomorphism class of $E_{b,\lambda}$ as a $\mathrm{Hod}_{\overline{\mathbb{Q}}}$-representation, for $b \in \mathcal{V}$. The atomic decomposition is the resulting *multiset*, indexed by $\operatorname{spec}(\boldsymbol{\kappa}_b)$.

Invariants of a coarse Hodge atom

Represent the atom α by a $\mathrm{Hod}_{\overline{\mathbb{Q}}}$-module E^α and put $E^\alpha_{\mathbb{C}} = E^\alpha \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. The cocharacter $\mu(u) = h_{\mathbb{C}}(u^{-1}, u)$ acts on $\mathrm{H}^{p,q}$ with weight $p - q$, defining

$$E^\alpha_{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}} (E^\alpha_{\mathbb{C}})_j, \qquad \mu(u)|_{(E^\alpha_{\mathbb{C}})_j} = u^j \,\mathrm{id}\,.$$

Definition (Numerical invariants)

$$\rho_\alpha = \dim_{\mathbb{C}} (E^\alpha_{\mathbb{C}})^{\mathrm{Hod}_{\mathbb{C}}},$$
$$P_\alpha(t) = \sum_{j \in \mathbb{Z}} \dim_{\mathbb{C}} (E^\alpha_{\mathbb{C}})_j \, t^j.$$

Thus $\dim E^\alpha = P_\alpha(1)$ and $\rho_\alpha \geq 1$ (KKPY). The coefficient of t^2 records Hochschild degree 2.

Two structural facts about atoms

Theorem (KKPY, irrationality criterion)

Let Z be smooth projective of dimension $n \geq 2$. If an atom of Z occurs in the atomic decomposition of no smooth projective variety of dimension at most $n-2$, then Z is irrational.

Theorem (KKPY)

If K_Z is nef, then Z has exactly one coarse Hodge atom (counted as a spectral factor).

For fourfolds

In a weak factorization from $\mathbb{P}^4$, only points, curves, and surfaces can contribute atoms from the blow-up centers.

Geometric setup and the fourfold criteria

Following the paper's standing assumptions, let

$$X \xhookrightarrow{j} Y \xhookrightarrow{\iota_Y} F = G/P, \quad Y = \mathscr{Z}(F, \mathcal{E}), \quad X = \mathscr{Z}(F, \mathcal{E} \oplus L),$$

where $L = \mathcal{O}_F(1, \dots, 1)$ and X is a smooth hyperplane section of the smooth Fano fivefold Y.

Standing assumptions (H0)–(H3)

- $\mathrm{Pic}(F) \xrightarrow{\sim} \mathrm{Pic}(Y) \xrightarrow{\sim} \mathrm{Pic}(X)$, and likewise for numerical curve lattices;
- $\iota_Y^* : \mathrm{H}^k(F, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^k(Y, \mathbb{Q})$ for $k < 4$;
- $\mathrm{H}^4(Y, \mathbb{Q})$ has Hodge type $(2, 2)$; Y is Fano.

For the criteria, also assume X Fano and $h^{3,1}(X) = 1$.

Which cohomology does the matrix see?

Write $\iota = \iota_Y \circ j$ and distinguish

$$\mathrm{H}^*_Y(X) = j^*\mathrm{H}^*(Y,\mathbb{Q}), \quad \mathrm{H}^*_F(X) = \iota^*\mathrm{H}^*(F,\mathbb{Q}), \quad W = \mathrm{H}^4(X)_{\mathrm{van}}.$$

At a small point b, Theorem 5.35 of the paper gives

$$\mathrm{H}^*(X,\Bbbk) = \mathrm{H}^*_Y(X)_\Bbbk \oplus W_\Bbbk, \qquad \boldsymbol{\kappa}_b = B_b \oplus \lambda_{\mathrm{van}}(b)\,\mathrm{id}_W\,.$$

Monodromy-fixed block

B_b is the action of $c_1(TX)\star_{\mathrm{small},b}$ on $\mathrm{H}^*_Y(X)_\Bbbk$.

Flag-ambient block

A_b is the pairing-orthogonal projection of the operator to $\mathrm{H}^*_F(X)_\Bbbk$.

Ambient completeness (AC)

If $\mathrm{H}^4(F,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^4(Y,\mathbb{Q})$, then $\mathrm{H}^*_F(X) = \mathrm{H}^*_Y(X)$ and $A_b = B_b$.

The algebraic-multiplicity criterion

Setup

Assume (H0)–(H3), X a Fano fourfold, $h^{3,1}(X) = 1$, and X Hodge general, i.e., $\mathrm{H}^4(X)^{\mathrm{Hdg}} = j^*\mathrm{H}^4(Y, \mathbb{Q})$.

Theorem

If every eigenvalue of the monodromy-fixed matrix $B^{\mathrm{alg}}(1)$ has algebraic multiplicity at most 2, then X is irrational. Under (AC), the condition may be checked on the computable matrix $A^{\mathrm{alg}}(1)$.

The Jordan-defect criterion: a K3 center

Assume (H0)–(H3), X Fano, and $h^{3,1}(X) = 1$, without Hodge generality.

Theorem

Suppose every eigenvalue of $B^{\mathrm{alg}}(1)$ has Jordan defect

$$m_a^Y(\lambda) - m_g^Y(\lambda) \leq 1.$$

If X is rational, then every weak factorization $X \dashrightarrow \mathbb{P}^4$ contains a smooth surface center whose minimal model is a projective K3 surface.

Under (AC), the defect condition may be checked on $A^{\mathrm{alg}}(1)$.

What must actually be computed?

Choose a homogeneous basis $\{\phi_i\}$ of $\mathrm{H}^*(X, \mathbb{Q})$ with Poincaré-dual basis $\{\phi^i\}$. Then

$$c_1(TX) \star_{\mathrm{small}} \phi_j = \sum_{\beta \in \mathsf{NE}(X)} \sum_i q^\beta \langle c_1(TX), \phi_j, \phi^i \rangle_{0,\beta}^X \, \phi_i.$$

For $\beta \neq 0$, the divisor axiom reduces the three-point invariants:

$$\langle c_1(TX), \phi_j, \phi^i \rangle_{0,\beta}^X = \left(\int_\beta c_1(TX) \right) \langle \phi_j, \phi^i \rangle_{0,\beta}^X.$$

The dimension constraint and Fano positivity leave only finitely many effective curve classes in each matrix entry.

Recovering the ambient matrix

Choose a basis $\phi_i = \iota^* \sigma_i$ of $\mathrm{H}^*_F(X)$ with nondegenerate restricted Poincaré pairing. Set

$$G_{ij} = \int_X \phi_i \phi_j, \qquad C_{ij}(q) = \sum_{\beta \in \mathsf{NE}(X)} q^\beta \langle \phi_i, c_1(TX), \phi_j \rangle^X_{0,\beta}.$$

With column vectors representing inputs, the projected operator satisfies

$$G\, A(q) = C(q), \qquad A(q) = G^{-1} C(q).$$

$A(q) = \mathrm{pr}_F \circ \big(c_1(TX) \star_{\mathrm{small}} (-)\big)|_{\mathrm{H}^*_F(X)}$. Curve classes in the flag-ambient computation are pushed forward by ι_*.

Localization replaces a moduli-space integral

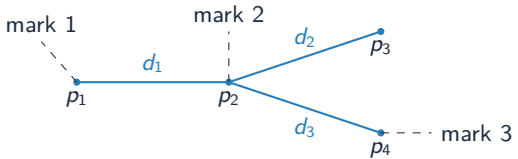
For the flag variety F, the primary invariants are

$$\langle \gamma_1, \dots, \gamma_m \rangle_{0,\beta}^{F} = \int_{[\overline{\mathcal{M}}_{0,m}(F,\beta)]^{\mathrm{vir}}} \prod_{k=1}^{m} \mathrm{ev}_k^* \, \gamma_k .$$

A maximal torus $\mathbb{T}$ acts on F and hence on stable maps. Virtual localization gives

$$\int_{[\overline{\mathcal{M}}]^{\mathrm{vir}}} \eta = \sum_{\Gamma} \int_{[F_\Gamma]^{\mathrm{vir}}} \frac{\eta|_{F_\Gamma}}{e_{\mathbb{T}}\big(N_\Gamma^{\mathrm{vir}}\big)} .$$

A fixed stable map is a decorated tree



- Vertices: fixed-point labels $p_{w_v} \in F^{\mathbb{T}}$ and markings mk.
- Edges: invariant rational curves with positive covering degrees d_e.
- Degree condition: $\beta = \sum_e d_e[C_e]$; $n_v = \mathrm{val}(v) + |\mathrm{mk}^{-1}(v)|$.

Contracted components contribute $\overline{\mathcal{M}}_{0,n_v}$ for $n_v \geq 3$; covered edges contribute finite automorphism factors.

Paper: Definition 3.11; Proposition 3.13

Flag varieties: the data we need

Let G be connected, simply connected, and semisimple, with $\mathbb{T} \subset B \subset P$. Put $F = G/P$.

Geometry	**Root-theoretic description**
Fixed points	W/W_P, represented by W_F
Cohomology	$\mathrm{H}^*(F, \mathbb{Q}) = \bigoplus_{w \in W_F} \mathbb{Q}\sigma_w$
Tangent weights at p_w	$\{-w\alpha : \alpha \in R_F^+\}$
Invariant curves	Root-labelled edges between fixed points
Curve classes	Reduced coroot lattice $Q_F^\vee$

$$R_F^+ = R^+ \setminus R_P^+, \quad \dim F = |R_F^+|, \quad c_1(TF) = \sum_{\alpha \in R_F^+} \alpha.$$

Passing to complete intersections

Let $\mathcal{E}$ be a globally generated homogeneous bundle on F, and $X = \mathscr{Z}(F, \mathcal{E})$ its smooth regular zero locus.

$$\dim X = \dim F - \operatorname{rk} \mathcal{E}, \qquad c_1(TX) = \iota^*\big(c_1(TF) - c_1(\mathcal{E})\big).$$

Convexity gives the vector bundle $\mathcal{E}_{0,m,\beta} = \pi_* f^* \mathcal{E}$ over $\overline{\mathcal{M}}_{0,m}(F, \beta)$.

$$\sum_{\substack{\beta' \in \mathrm{NE}(X) \\ \iota_* \beta' = \beta}} \langle \iota^*\gamma_1, \ldots, \iota^*\gamma_m \rangle_{0,\beta'}^{X} = \int_{[\overline{\mathcal{M}}_{0,m}(F,\beta)]^{\mathrm{vir}}} e(\mathcal{E}_{0,m,\beta}) \prod_{k=1}^{m} \mathrm{ev}_k^* \gamma_k.$$

The same localization formula applies with one additional Euler-class factor (Theorem 3.28 of the paper).

The accompanying software

GEOATOMS · `gwflags`

Root system + parabolic + bundle $\longrightarrow$ localization $\longrightarrow$ quantum matrix and characteristic polynomial.

The website provides the catalogue, software, references, and inputs for reproducing the examples.

`https://leonardofcavenaghi.github.io/atomicdecompositions/`

Example	**Input**
Cubic fourfold	A_5, keep 1; bundle $\mathcal{O}(3)$
Ordinary Gushel–Mukai	A_4, keep 2; bundle $\mathcal{O}(1) \oplus \mathcal{O}(2)$

Example: a cubic fourfold

$X = \mathscr{Z}(\mathbb{P}^5, \mathcal{O}(3))$, with $h = c_1(\mathcal{O}_X(1))$ and $c_1(TX) = 3h$.

$$A(q) = \begin{pmatrix} 0 & 0 & 18q & 0 & 0 \\ 3 & 0 & 0 & 45q & 0 \\ 0 & 3 & 0 & 0 & 18q \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix}, \qquad \text{basis } (1, h, h^2, h^3, h^4).$$

$$\chi_{A(q)}(\lambda) = \lambda^2(\lambda^3 - 729q).$$

For Theorems 5.47–5.48, realize X as a hyperplane section of a smooth cubic fivefold $Y \subset \mathbb{P}^6$. Lefschetz gives (AC), so $B^{\mathrm{alg}}(1) = A^{\mathrm{alg}}(1)$.

Reading the cubic spectrum

At the algebraic augmentation $q = 1$:

$$\operatorname{spec}(A^{\mathrm{alg}}(1)) = \{0, 9, 9\zeta, 9\zeta^2\}, \qquad \zeta = e^{2\pi \mathrm{i}/3}.$$

$m_a(0) = 2$, $m_g(0) = 1$; all nonzero eigenvalues are simple.

The vanishing Hodge structure:

$$(h^{3,1}_{\mathrm{van}}, h^{2,2}_{\mathrm{van}}, h^{1,3}_{\mathrm{van}}) = (1, 20, 1).$$

$\lambda_{\mathrm{van}} = 0$. For Hodge-general cubics, W contains no rational Hodge classes.

Theorem 5.47 proves irrationality of every Hodge-general cubic; at $\lambda = 0$ one has $\nu = 1$ and $\rho = 2$.

Theorem 5.48: any rational smooth cubic would have a K3 surface center in *every* weak factorization.

Example: ordinary Gushel–Mukai fourfolds

$$X = \mathscr{Z}\big(\mathrm{Gr}(2,5), \mathcal{O}(1) \oplus \mathcal{O}(2)\big), \qquad c_1(TX) = 2h.$$

Take $Y = \mathscr{Z}(\mathrm{Gr}(2,5), \mathcal{O}(2))$. Lefschetz gives (AC), and the fixed block has dimension 6.

$$\chi_{A(q)}(\lambda) = \lambda^2(\lambda^4 - 176q\lambda^2 - 256q^2).$$

$$m_a(0) = 2, \qquad m_g(0) = 2, \qquad \lambda_{\mathrm{van}} = 0.$$

The zero eigenspace is semisimple. Theorem 5.47 proves irrationality for Hodge-general ordinary GM fourfolds; Theorem 5.48 imposes a K3 center on any rational member.

Compare Benedetti–Manivel–Perrin; paper, Example 6.5

Küchle (c5)

$$F = \mathrm{Gr}(3,7), \quad Y = \mathscr{Z}(F, \Lambda^2\mathcal{U}^\vee \oplus \Lambda^3\mathcal{Q}), \quad X = Y \cap H.$$

$$b_4(F) = 2, \qquad b_4(Y) = 4, \qquad h^{2,2}(X) = 24.$$

The 6×6 flag-ambient matrix misses two fixed $(2,2)$-classes. Its spectrum cannot replace that of B.

The numerical irrationality criterion

For Hodge-general X,

$$b_4(Y) \leq h^{2,2}(X) - 20 \qquad \text{since} \qquad 4 = 24 - 20.$$

Thus the paper's numerical criterion proves irrationality without computing the Y-block.